ISSN 2229-5518

ON MULTISET FUNCTIONS

Gambo Jeremiah Gyam[1] and Y. Tella[2]

1. Nassarawa State University, Keffi, Nigeria. E-mail: ggjerry016@gmail.com.
2. Kaduna State University, Kaduna, Nigeria. E-mail: yot2002@yahoo.com.

Abstract:

This paper introduces and presents multiset function from a very unique and different way. It is build on the studies from previous research works. Some of the basic principles and properties of functions are studied in multiset context such as injection, surjection, bijection, identity, and constant functions. The composition of functions is studied. Similarity and dominance relations are also studied among others.

Key Words: Multiset, Multiset Functions, Composition of multiset functions, Similarity and Dominance relation in multiset.

1. Introduction

Multiset as a new paradigm shift of science and mathematics which is fast moving to take over the affairs of the world for its ability to allow its elements to repeat itself in a set. These assertions were against the Cantor's resolutions that no element should be repeated in a set. He stated that each element should be distinct and unique. However, in science and in ordinary life concept found this found faulty. In the physical world there is enormous repetition. For instance, there are many hydrogen atoms, many water molecule, many strands of identical DNA e.t.c.

Multiset (mset for short) then is a set in which repetition is allowed. In fact, the term was first suggested by N.G. De. Brujin to Knuth in a private communication as the generalization of the crisp theory. Thus every set is a mset, but the reverse is not true [4]. Other works on mset were found in [3], [5],[9], [12]. [13] and [14]. Early researchers gave several names to mean mset, such as bags, heap, fireset (finitely repeated set) [15].

As mentioned earlier, elements are allowed to repeat in an mset. The number of copies ([14], p.3) prefers to call it 'Multiples' of an element in a mset which may be finite or infinite, positive or negative. Their multiplicities jointly determined its cardinality.

Hickman, [14] was among the earliest mathematicians to begin to work on functions between one mset and another. Blizard, [1] in his doctoral thesis axiomatize the functions between two msets. He worked on the special types of such functions e.g injection, surjection and bijection. Girish and John in [12] from a different perspective study the functions between two msets using the principle of relations between two msets. They extended the study to include the study of constant and identity mset functions.

In this paper, we redefine mset functions carrying sufficient ingredients to make the mset function viable to further studies. In section two, some of the basic definitions, operations and notations of mset functions are presented. Some algebraic properties of our approach are studied in section three and section four characterizes the summary of our findings and directions for further studies.

2.1 Preliminary definitions and notations

Definition 2.1.1[1]**.** An mset A over the set X can be defined as a function $C_A: X \to \mathbb{N} = \{0,1,2, ...\}$ where the value $C_A(x)$ denote the number of times or multiplicity or count function of x $in\ A$. For example, Let $A = [x, x, x, y, y, y, z, z]$, then $C_A(x) = 3, C_A(y) = 3, C_A(z) = 2$. $[C_A(x) = 0 \Leftrightarrow x \notin A]$. The mset M over the set X is said to be empty if $C_M(x) = 0\ for\ all\ x \in X$. We denote the empty mset by $\emptyset$. Then $C_\emptyset(x) = 0, \forall\ x \in X.$ if $C_A(x) > 0, then\ x \in A$. We denote the class of all finite msets M over the set X by $M(X)$ throughout the study. If $C_A(x) = n$ then the membership of x in A can be denoted by $x \in^n A$, meaning x belong to A exactly n times.

Definition 2.1.2[1]**:** The cardinality of a mset M denoted $|M|$ or $card(M)$ is the sum of all the multiplicities of its elements given by the expression $|M| = \sum_{x\in X} c_A(x)$.

Note: Presentation of mset on paper work became a challenged as every researcher has his thought in that aspect . However the use of square brackets was adopted in ([1], [9],[11]) to represent an mset and ever since then it has become a standard. For example if the multiplicity of the elements x, y and z in an mset M are 2,3 and 2 respectively, then the mset M can be represented as $M = [x, x, y, y, y, z, z,]$, others may put it like $[x, y, z]_{2,3,2}$ or $[x^2, y^3, z^2]$ or $[x2, y3, z2]\ or\ [2/x , 3/y , 2/z]$ depending on one's taste or expediencies. But for conveniences sake, curly bracket can be used instead of the square bracket.

Definition 2.1.3[2]: Let M be an mset drawn from a set X. The support set of M denoted by M^* is a subset of X given by $M^* = \{x \epsilon X: C_M(x) > 0\}$. M^* is also called root set.

Definition 2.1.4[1](Equal msets): Two msets $A, B \in M(X)$ are said to be equal, denoted $A = B$ if and only if for any objects $x \in X, C_A(x) = C_B(x)$. This is to say that $A = B$ if the multiplicity of every element in A is equal to its multiplicity in B and conversely.

Note that $A = B \Longrightarrow A^* = B^*$, though the converse need not hold. For example, let $A = [a, a, b, b, c]\ and\ B = [a, a, b. b, b, c, c]\ where\ A^* = B^* = \{a, b, c\}\ but\ A \neq B.$

Definition 2.1.5[1](Submultiset): Let $A, B \in M(X)$. A is a submultiset (submset for short) of B, denoted by $A \subseteq B\ or\ B \supseteq A$, if $C_A(x) \leq C_B(x)\ for\ all\ x \in X$. Also if $A \subseteq B\ and\ A \neq B$, then A is called proper submset of B denoted by $A \subset B$. In other words $A \subset B$ if $A \subseteq B\ and\ there\ exist\ at\ least\ an\ x \in X$ such that $C_A(x) < C_B(x)$. We assert that a mset B is called the parent mset in relation to the mset A.

Theorem 2.1.6[11]: Let $M, N \in M(X)$, $M \subseteq N \Rightarrow M^* \subseteq N^*$.

Note that: For any two msets $A, B \in M(X)$, $A = B$ if and only if $A \subseteq B$ and. $B \subseteq A$.

Definition**.** 2.1.7[1](Regular or Constant mset): An mset A over the set X is called regular or constant if all its elements are of the same multiplicities, i.e for any $x, y \in A\ such\ that\ x \neq y, C_A(x) = C_A(y)$.

Definition 2.1.8 [17](Power mset): Let $A \in M(X)$. The power mset of A, denoted $\wp(A)$, is defined as the mset of all submsets of A i.e $\wp(A) = \{m/p \mid p \subseteq A\ and\ p \in^n \wp(A)\}$. For instance if $A = [x, y]_{2,1} = [x, x, y]$. Then $\wp(A) = \left[\emptyset, \{x\}, \{x\}, \{x\}_2, \{y\}, \{x, y\}, \{x, y\}, [x, y]_{2,1}\right]$.

In this case the cardinality of $\wp(A)$ is given by $Card(\wp(A)) = 2^{Card(A)} = 2^3 = 8$, for any mset A.

For any $N \subseteq M$ such that $N \neq \emptyset$.

Now $N \in^k \wp(M)$ $if\ and\ only\ if$ $k = \prod_z \binom{|M|_z}{|N|_z}$. Where $\prod_z$ is the product taken over distinct elements z of the mset N . $|M|_z = m\ iff\ z \in^m M\ and\ |N|_z = n\ iff\ z \in^n N$.

Note that $\binom{|M|_z}{|N|_z} = \binom{m}{n} = \frac{m!}{n!(m-n)!}$.

We denote the root set of $\wp(M)$ by $\wp^*(M)$.

Definition 2.1.9[17](Power set of an mset): Let $M \in M(X)$, the power set of M is just the root set $\wp^*(M)$.

Example 2.1.10: Let $M = \{6/x\,, 3/y\}$ be an mset and let $\wp(M)$ denote the power mset, if $\{3/x\}$ is a member of $\wp$(M), then $\{3/x\}$ repeats $k = \binom{6}{3} = 20\ times$. Also, if $\{4/x\,, 2/y\}$ is a member of $\wp$(M), then $\{4/x\,, 2/y\}$ repeats $k = \binom{6}{4}\binom{3}{2} = 45\ times$.

Theorem 2.1.11[17](Cardinality of power set): Let $M \in M(X)$ such that $M = \{{}^{m_1}/_{x_1}, {}^{m_2}/_{x_2}, \dots, {}^{m_n}/_{x_n}\}$, then $Card(\wp^*(M)) = \prod_{i=1}^{n}(1+m_i)$.

Definition 2.1.12[17](Whole submset): A submset N of M is a whole submset of M

if $C_N(x) = C_M(x)\ \forall\, x \in N$.

Definition 2.1.13[17](Partial Whole Submset): A submset N of M is a partial whole submset of M if there exist an element $x \in N$ such that $C_N(x) = C_M(x)$.

Definition 2.1.14[17](Full Submset): A submset N of M is full submset if $M^* = N^*$

Example 2.1.15: Let $M = \{2/x\,, 3/y\,, 5/z\}$ be an mset. The following are some of the submset which are whole submsets, partial whole submset and full submets.

(a) A submset $\{2/x\,, 3/y\}$ is a whole submset and partial whole submset of M but it is not full submset of M.
(b) A submset $\{1/x\,, 3/y\,, 2/z\}$ is a partial whole submset and full submset of M but it is not a whole submet of M.
(c) A submset $\{1/x\,, 3/y\}$ is a partial whole submset of M which is neither a whole submet nor full submset of M.

Definition 2.1.16[17, 15] (Power whole mset): Let $M \in M(X)$ be an mset. The power whole mset of M denoted by $PW(M)$ is defined as the set of all whole submsets of M. The cardinality of the support set $PW(M)$ is 2^n where n is the cardinality of the support set M^*, i.e $n = |\, M^*\, |$.

Definition 2.1.17[17] (Power full mset): Let $M \in M(X)$ be an mset. Then the power full mset of M denoted, $PF(M)$, is defined as the set of all full submsets of M. The cardinality of $PF(M)$ is the product of the counts of the elements in M.

That is $PF(M) = \{y/\ y \subseteq M\}$.

Examples 2.1.18: Let $M = \{2/x\,, 3/y\}$ be a mset. Then $PW(M) = \{\{2/x\}, \{3/y\}, M, \emptyset\}$ and $PF(M) = \{\{2/x\,, 1/y\}, \{2/x\,, 2/y\}, \{2/x\,, 3/y\}, \{1/x\,, 1/y\}, \{1/x\,, 2/y\}, \{1/x\,, 3/y\},\}$.

Definition 2.1.19 [9] ($\wedge$ and $\vee$ notations): The notations $\wedge$ and $\vee$ denote the minimum and maximum operator respectively, for instance;

$C_A(x) \wedge C_A(y) = min\{C_A(x), C_A(y)\}\ and\ C_A(x) \vee C_A(y) = max\{C_A(x), C_A(y)\}$.

2.2 mset operations.

Definition 2.2.1[9] (msets union): Let $A, B \in M(X)$. The union of $A\ and\ B$ denoted $A \cup B$ is the mset defined by $C_{A\cup B}(x) = max\{C_A(x), C_B(x)\}$,

Definition 2.2.2[9] (msets intersection) Let $A, B \in M(X)$.The intersection of two mset A and B denoted by $A \cap B$, is the mset for which

$C_{A\cap B}(x) = min\{C_A(x), C_B(x)\}\ \ \forall\, x \in X$.

Definition 2.2.3[9] (mset addition): Let $A, B \in M(X)$. The direct sum or arithmetic addition of A and B denoted by $A + B$ or $A \uplus B$ is the mset defined by

$C_{A+B}(x) = C_A(x) + C_B(x)\ \ \forall\, x \in X$.

Note that $|\ A \uplus B\ | = |\ A \cup B\ | + |\ A \cap B\ |$.

Definition 2.2.4[9] (mset difference): Let $A, B \in M(X)$, then the difference of B from A, denoted by $A - B$ is the mset such that $C_{A-B}(x) = max\{C_A(x) - C_B(x), 0\}\ \forall\, x \in X$. If $B \subseteq A$, then

$C_{A-B}(x) = C_A(x) - C_B(x)$.

It is sometimes called the arithmetic difference of B from A. If $B \nsubseteq A$ this definition still holds. It follows that the deletion of an element x from an mset A give rise to a new mset $A' = A - x$ such that $C_{A'}(x) = max\{C_A(x) - 1, 0\}$.

Definition 2.2.5[8] (mset symmetric difference): Let X be a set and $A, B \in M(X)$ Then the symmetric difference, denoted $A\Delta B$, is defined by $C_{A\Delta B}(x) = |C_A(x) - C_B(x)|$.

Note that $A\Delta B = (A - B) \cup (B - A)$.

Definition 2.2.6[8] (mset complement): Let $G = \{A_1, A_2, \dots, A_n\}$ be a family of finite msets generated from the set X. Then, the maximum mset Z is defined by $C_Z(x) = max_{A\in G} C_A(x)$ for all $A \in G$ and $x \in X$. The Complement of an mset A, denoted by $\overline{A}$, is defined:

$\bar{A} = Z - A\ such\ that\ \ C_{\bar{A}}(x) = C_Z(x) - C_A(x), for\ all\ x \in X$.

Note that $A_i \subseteq Z \; for \; all \; i$.

Definition 2.2.7[8] (Multiplication by Scalar): Let $A \in M(X)$, then the scalar multiplication denoted by $b.A$ is defined by $C_{b.A}(x) = b.C_A(x), \; and \; b \in \{1,2,3, \ldots\}$.

Definition 2.2.8[8] (Arithmetic Multiplication): Let $A, B \in M(X)$, then the Arithmetic Multiplication denoted by $A.B$ is defined by $C_{A.B}(x) = C_A(x).C_B(x) \;\; \forall \, x \in X$.

Definition 2.2.9[7] (Raising to an Arithmetic Power): Let $A \in M(X)$, then A raised to power n denoted by A^n **is** defined:

$C_{A^n}(x) = \left(C_A(x)\right)^n \; for \; n \in \{0,1,2,3, \ldots\}$ and $C_A(x) > 0$.

Proposition 2.2.10: For any $A \neq \emptyset$ such that $A \in M(X)$, then $(A^n)^* = A^*$ for $n \in \{0,1,2 \ldots\}$

Proof: Now let $x \in (A^n)^*$, then $C_{A^n}(x) > 0$. That is $\left(C_A(x)\right)^n > 0$.

$\left(C_A(x)\right)^n > 0 \;\Rightarrow C_A(x) > 0$ (by definition 2.1.3). Thus $x \in A^*$

In particular $(A^n)^* \subseteq A^*$ (i)

Similarly let $y \in A^*$. Then $C_A(y) > 0$ (by definition 2.1.3)

Clearly $C_A(y) > 0 \Rightarrow C_{A^n}(y) > 0 \; for \; n = \{0,1,2 \ldots\}$

Now $\left(C_A(y)\right)^n > 0 \;\Rightarrow C_{A^n}(y) > 0$

In particular $y \in A^*$ (by definition 2.2.9)

Thus $A^* \subseteq (A^n)^*$ (ii)

Now from (i) and (ii) above, it is clear that $A^* = (A^n)^*$

2.3 mset functions.

Definiton 2.3.1: Let X be a set and let $A, B \in M(X)$. We defined the mset function $f: A \to B$ as just the function $f: A^* \to B^*$ such that for any $x \in X$, $C_{f(A)}\big(f(x)\big) = C_A(x)$. Where

$$f(A) = \left\{\frac{m_i}{f(x_i)} : \; x \in A, \; m_i = \; C_{f(A)}\big(f(x_i)\big) = C_A(x_i) \right\}.$$

Example 2.3.2: Let $X = \{x, y\}, Y = \{z_1, z_2, z_3\}$ be a sets. Also, let $A = \{x, y\}_{2,3}$ and $B = \{z_1, z_2, z_3\}_{3,4,5}$, the function $f: A \to B$ is an mset function if we define $f(x) = z_1 \; and \; f(y) = z_2 \; \forall \, x \in X$. Since $f: A^* = \{x, y\} \to \{z_1, z_2\} = B^*$ is a function and $f\big(\{x, y\}_{2,3}\big) = \{z_1, z_2\}_{2,3} \subseteq \{z_1, z_2, z_3\}_{3,4,5}$.

Hence f is a mset function and $f(A) = \{z_1, z_2\}_{2,3}$.

Hence f is a mset function and $f(A) = \{z_1, z_2\}_{2,3}$

Definition 2.3.3(Constant mset function): Let $A, B \in M(X)$. The constant mset function $f: A \to B$ is just the mset function $f: A \to B$ such that $f: A^* \to B^*$ is constant.

Definition 2.3.4(Identity mset function): Let $A, B \in M(X)$. The identity mset function $f: A \to B$ is just the mset function $f: A \to B$ such that $f: A^* \to B^*$ is identity.

Definition 2.3.5(Injective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be injective if

(i) $f: A^* \to B^*$ is injective and

(ii) $\forall x(x \in A^* \Rightarrow C_A(x) \leq C_B(f(x))$.

Example 2.3.6: Let $X = \{1, -1, i, -i\}$ be set, where $i = \sqrt{-1}$. Let $A = \{1, -1\}_{2,1}$, $B = \{1, -1, i, -i\}_{5,3,2,2}$ be msets. We defined our function as $f(x) = x$, $\forall x \in X$. $f(1) = 1$ and $f(-1) = -1$.

Clearly $f: A^* = \{1, -1, i, -i\} \to \{1, -1\} = B^*$ is a injective function.

and $C_A(1) = 2 \leq C_B(f(1)) = 5$, $C_A(-1) = 1 \leq C_B(f(-1)) = 3$.

Hence f is an injective mset function.

Definition 2.3.7(Surjective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be surjective if

(i) $f: A^* \to B^*$ is surjective and

(ii) $\forall x(x \in A^* \Rightarrow C_A(x) \geq C_B(f(x))$.

Example 3.3.8: Let $X = \{1, -1, i, -i\}$ be set, where $i = \sqrt{-1}$. Let $A = \{1, -1, i, -i\}_{4,4,3,3}$, $B = \{1, -1\}_{2,1,}$ be msets. We defined our function as $f(x) = x^2$, $\forall x \in X$. $f(1) = 1^2 = 1$ and

$f(-1) = (-1)^2 = 1, f(i) = (i)^2 = -1 \; and \; f(-i) = (-i)^2 = -1$.

Clearly $f: A^* = \{1, -1, i, -i\} \to \{1, -1\} = B^*$ is a surjective function.

Thus $C_A(1) = 4 \geq C_B(f(1)) = C_B(1) = 2$, $C_A(-1) = 4 \geq C_B(f(-1)) = C_B(1) = 2$,

$$C_A(i) = 3 \geq C_B(f(i)) = C_B(-1) = 1, and \;\; C_A(-i) = 3 \geq C_B(f(-i)) = C_B(-1) = 1.$$

Hence f is a surjective mset function.

Definition 2.3.9(Bijective mset function): Let $A, B \in M(X)$. The mset function $f: A \to B$ is said to be a bijective if

(i) $f: A^* \to B^*$ is bijective and

(ii) $\forall x(x \in A^* \Rightarrow C_A(x) = C_B(f(x))$.

Example 2.3.10: Let $X = \{1,2,3,8,27\}$ be a set. Let $A = \{1,8,27\}_{2,4,3}$, $B = \{1,2,3\}_{2,4,3,}$ be msets. We defined our function as $f(x) = \sqrt[3]{x}$, $\forall x \in X$. $f(1) = \sqrt[3]{1} = 1$, $f(8) = \sqrt[3]{8} = 2$ and $f(27) = \sqrt[3]{27} = 3$.

Clearly $f: A^* = \{1,8,27\} \to \{1,2,3\} = B^*$ is a bijective function.

Thus $C_A(1) = 2 = C_B f(1) = 2$, $C_A(8) = 4 = C_B f(8) = 4$, $C_A(27) = 3 = C_B f(27) = 3$.

Hence f is a bijective mset function.

Definition 2..3.11(Inverse mset function): Let $A, B \in M(X)$. The inverse of the mset function $f: A \to B$ denoted $f^{-1}: B \to A$ is just the function $f^{-1}: B^* \to A^*$.

Definition 2.3.12(Composition of mset function): Let A, B and $C \in M(X)$. We defined the composition of mset function of $f: A \to B$ and $g: B \to C$ denoted as $gof: A \to C$ as just the composition $gof: A^* \to C^*$, such that $C_{gof(A)}\big(gof(x)\big) = C_A(x)$.

Definition 2.3.13(Similarity Relation): Let $A, B \in M(X)$. A and B are said to be similar denoted $A \sim B$ if there exist a bijection between A and B.

Definition 2.3.14(Dominance Relation): Let $A, B \in M(X)$, then A is dominated by B denoted $A \preccurlyeq B$ if there exist an injection between A and B.

3. Some Related Results.

Proposition 3.1: Let $A, B \in M(X)$. The mset function $f: A \to B$ is a bijective mset function, if and only if $f^{-1}: B \to A$ is bijective.

Proof: Given that $f: A \to B$ is bijective, then.

$f: A^* \to B^*$ is bijective and (by definition 2.3.8) (i)

$\forall x (x \in A^* \Rightarrow C_A(x) = C_B\big(f(x)\big)$ and (ii)

Since from (i) it follows that $f^{-1}: B^* \to A^*$ is bijective (iii)

Now we show that $\forall y$ ($\forall$ $y \in B^* \Rightarrow C_B(y) = C_A\big(f^{-1}(y)\big)$

And for every $y \in B^*$ there exist $x \in A^*$ such that $f(x) = y$. (iv)

In particular, $x = f^{-1}(y)$ (from (i) above) (v)

And from (v), we have $C_A(x) = C_A\big(f^{-1}(y)\big)$ and

$C_B(f(x)) = C_A\big(f^{-1}(y)\big)$ (vi)

In particular, $C_B(y) = C_A\big(f^{-1}(y)\big)$ [from (iv) above] (vii)

Thus from (iii) and (vii) above, it is clear that

$f^{-1}: B \to A$ is bijective. (by definition 2.3.8)

Conversely, let $f^{-1}: B \to A$ be bijective, we show that $f: A \to B$ is bijective.

Since $f^{-1}: B^* \to A^*$ is bijective (by definition 2.3.8) (viii)

In particular $(f^{-1})^{-1}: A^* \to B^*$ is bijective. (ix)

Also, $\forall y$ ($\forall$ $y \in B^* \Rightarrow C_B(y) = C_A\big((f^{-1}(y)\big)$ by definition 2.3.8 (x)

But for all $x \in A^*$ there exist $y \in B^*$ such that

$f^{-1}(y) = x$ (xi)

In particular $y = f(x)$ (xii)

Now substituting (xi) and (xii) in (x) above, we have

$C_B(f(x)) = C_A(x) \;\; x \in A^*$.

In particular, we have

$\forall x(x \in A^* \Rightarrow C_A(x) = C_B\big(f(x)\big)$ (xiii)

Thus from (ix) and (xiii), we have $f: A \to B$ bijective.

Proposition 3.2: Let $A, B \in M(X)$. If the mset function $f: A \to B$, is injective, surjective and bijective respectively, then $f(A) \subseteq B,\; B \subseteq f(A)$ *and* $f(A) = B$ respectively.

Proof: Let the mset function $f: A \to B$ be injective, then

$f: A^* \to B^*$ is injective (i)

Then $\forall x(x \in A^* \Rightarrow C_A(x) \leq C_B\big(f(x)\big)$ (ii)

$C_A(x) = C_{f(A)}\big(f(x)\big)$ by definition (2.3.1 and 2.3.5) (iii)

Now substituting (iii) in (ii), we have $C_{f(A)}(f(x)) \leq C_B\big(f(x)\big)$ (iv)

Thus $f(A) \subseteq B$.(from iv)

Again, let the mset function $f: A \to B$ be surjective, then

$C_A(x) = C_{f(A)}\big(f(x)\big) \;\forall\; x \in A^*$ (by definition 2.3.1) (v)

$\forall x(x \in A^* \Rightarrow C_A(x) \geq C_B\big(f(x)\big)$ by definition 2.3.7 (vi)

Substituting (v) in (vi) we get $C_{f(A)}(f(x)) \geq C_B\big(f(x)\big)$ (vii) and

$B \subseteq f(A)$. (from vii)

Assuming $f: A \to B$ be a bijective mset function, then it is both injective and surjective

In particular

$f(A) \subseteq B$ (viii) and

$B \subseteq f(A)$ (ix) from the above results.

Thus, from (viii) and (ix) above, it is clear that

$f(A) = B$.

Proposition 3.3: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are injective mset functions, then $gof: A \to C$ is injective.

Proof: Supposing $f: A \to B$ and $g: B \to C$ are injective mset functions, then

$f: A^* \to B^*$ and $g: B^* \to C^*$ (i) are injective and

$\forall x(x \in A^* \Rightarrow C_A(x) \leq C_B(f(x))$ (ii)

$\forall y(y \in B^* \Rightarrow C_B(y) \leq C_C(g(y))$ (iii)

Since $f(x) \in B^*$, we have

$\forall x(x \in A^* \Rightarrow C_A(x) \leq C_B(f(x)) \leq C_C(g(f(x)))$(from (ii) and (iii))

In particular,

$\forall x(x \in A^* \Rightarrow C_A(x) \leq C_C(g\mathrm{o}f(x)))$ (iv)

But from (i) the composition

$g\mathrm{o}f: A^* \to C^*$ is injective (v)

Now from (iv) and (v) above, we have

$g\mathrm{o}f: A \to C$ injective as well.

Proposition 3.4: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are surjective mset functions, then $g\mathrm{o}f: A \to C$ is surjective.

Proof: Supposing $f: A \to B$ and $g: B \to C$ are surjective mset functions, then

$f: A^* \to B^*$ and $g: B^* \to C^*$ (i) are surjective and

$\forall x(x \in A^* \Rightarrow C_A(x) \geq C_B(f(x))$ (ii)

$\forall y(y \in B^* \Rightarrow C_B(y) \geq C_C(g(y))$ (iii)

Since $f(x) \in B^*$, we have

$\forall x(x \in A^* \Rightarrow C_A(x) \geq C_B(f(x)) \geq C_C(g(f(x)))$(from (ii) and (iii))

In particular,

$\forall x(x \in A^* \Rightarrow C_A(x) \geq C_C(g\mathrm{o}f(x)))$ (iv)

But from (i) the composition

$g\mathrm{o}f: A^* \to C^*$ is surjective (v)

Now from (iv) and (v) above, we have

$g\mathrm{o}f: A \to C$ also surjective.

Proposition 3.5: Let $A, B, C \in M(X)$, if $f: A \to B$ and $g: B \to C$ are bijective mset functions, then we show that $g\mathrm{o}f: A \to C$ is bijective.

Proof: Supposing $f: A \to B$ and $g: B \to C$ are bijective mset functions, then we show that $g\mathrm{o}f: A \to C$ is bijective.

Clearly from the hypothesis and proposition 3.2, $f(A) = B$ and $g(B) = C$.

Thus$gof(A) = g\big(f(A)\big) = g(B) = C$.

Hence $gof(A) = C$.

Proposition 3.6: Let $A \in M(X)$. The identity mset function $I_A: A \to A$ is bijective.

Proof: For any $A \in M(X)$, the identity mset function $I_A: A \to A$ is just the mset function $I_A: A \to A$ such that $I_A: A^* \to A^*$ is identity (by definition 2.3.4)

In particular, the identity mset function $I_A: A \to A$ is just the identity function $I_A: A^* \to A^*$ (i)

But the identity function $I_A: A^* \to A^*$ is bijective (ii)

Now we show that $\forall x(x \in A^* \Rightarrow C_A(x) = C_{I_A}\big(I_A(x)\big)$

Since $I_A(x) = x$, we have (by definition)

Thus,$C_A(x) = C_A\big(I_A(x)\big)$ (iii)

Thus from (ii) and (iii) above, it is clear that I_A is bijective.

Proposition 3.7: Let $A, B \in M(X)$ and let $f: A \to B$ be an mset function. If $A_1 \subseteq A$ and $A_2 \subseteq A$, Also $B_1 \subseteq B$ and $B_2 \subseteq B$ such that $f: A_1 \to B_1$ and $f: A_2 \to B_2$, are mset functions, then

(i) $A_1 \subseteq A_2 \Rightarrow f(A_1) \subseteq f(A_2)$
(ii) $f(A_1 \cup A_2) \supseteq f(A_1) \cup f(A_2)$
(iii) $f(A_1 \cap A_2) \subseteq f(A_1) \cap f(A_2)$
(iv) $f(A_1) \cup f(A_2) \subseteq f(A_1 \oplus A_2)$

(i) Proof : Supposing $A_1 \subseteq A_2$ then $C_{A_1}(x) \leq C_{A_2}(x) \;\forall\, x \in X$ (i) by definition 2.1.5

Since $f: A_1 \to B_1$ and $f: A_2 \to B_2$, are mset functions, then

$C_{A_1}(x) = C_{f(A_1)}\big(f(x)\big)$, $C_{A_2}(x) = C_{f(A_2)}(f(x)) \;\forall\, x \in X$. (ii)

In particular, from (i) and (ii) it implies that $C_{f(A_1)}(f(x)) \leq C_{f(A_2)}\big(f(x)\big) \;\forall\, x \in X$

Thus $f(A_1) \subseteq f(A_2)$.

(ii) Proof : Let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$.

$C_{A_1}(x) \leq \max\{C_{A_1}(x), C_{A_2}(x)\}$ and $C_{A_2}(x) \leq \max\{C_{A_1}(x), C_{A_2}(x)\}$,

thus $A_1 \subseteq A_1 \cup A_2$ and $A_2 \subseteq A_1 \cup A_2$ and

$$A_1 \subseteq A_1 \cup A_2 \;\Rightarrow f(A_1) \subseteq f(A_1 \cup A_2)$$

$$A_2 \subseteq A_1 \cup A_2 \;\Rightarrow f(A_2) \subseteq f(A_1 \cup A_2) \text{ from (i) above}$$

Therefore $f(A_1) \cup f(A_2) \subseteq f(A_1 \cup A_2)$

(iii) Proof : Let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$.

$C_{A_1}(x) \geq \min\{C_{A_1}(x), C_{A_2}(x)\}$ and $C_{A_2}(x) \geq \min\{C_{A_1}(x), C_{A_2}(x)\}$,

Thus $A_1 \supseteq A_1 \cap A_2$ and $A_2 \supseteq A_1 \cap A_2$.

$$A_1 \supseteq A_1 \cap A_2 \quad \Rightarrow f(A_1) \supseteq f(A_1 \cap A_2)$$

$$A_2 \supseteq A_1 \cap A_2 \quad \Rightarrow f(A_2) \supseteq f(A_1 \cap A_2) \text{ from (i) above}$$

Therefore $f(A_1) \cap f(A_2) \supseteq f(A_1 \cap A_2)$

(iv) Proof : Let $A_1, A_2 \subseteq A$ and $B_1, B_2 \subseteq B$.

$$A_1 \subseteq A_1 + A_2 \quad \Rightarrow f(A_1) \subseteq f(A_1 + A_2)$$
$$A_2 \subseteq A_1 + A_2 \quad \Rightarrow f(A_2) \subseteq f(A_1 + A_2)$$

Thus $f(A_1) \cup f(A_2) \subseteq f(A_1 + A_2)$.

Again if $A_1 \cap A_2 \subseteq A_1 + A_2$

Therefore $f(A_1 \cap A_2) \subseteq f(A_1 + A_2)$ from (i) above, then

$$f(A_1 \cap A_2) \subseteq f(A_1 + A_2) \supseteq f(A_1) \cup f(A_2)$$

Hence the result.

Proposition 3.8: Let $f: A \to B$ be a bijective mset function, then

(a) $f^{-1} \text{o} f = I_A$
(b) $f \text{o} f^{-1} = I_B$

Where $I_A: A \to A$ is iidentity on A.

(a) Proof: Assuming $f: A \to B$ a bijective mset function, then we have
$f: A^* \to B^*$ Bijective (i)
$\forall x (x \in A^* \Rightarrow C_A(x) = C_B(f(x)))$ (ii) by definition 2.3.9

$f^{-1}: B \to A$ is a bijective (from proposition 2.3.1) (iii)

In particular

$f^{-1}: B^* \to A^*$ is bijective (iv)

Now from (i) and (iv) above, we have

$f^{-1} \text{o} f: A^* \to A^*$ is identity in A^*

Thus $f^{-1} \text{o} f = I_A$ (by definition 2.3.4)

and similarly for (b)

Proposition 3.9: Let $A, B \in M(X)$. If $f: A \to B$ is bijective, then $(f^{-1})^{-1} = f$.

Proof: Let $f: A \to B$ be a bijective mset function, then $f^{-1}: B \to A$ is a bijective mset function(by proposition 3.1)

In particular $f: A^* \to B^*$ and $f^{-1}: B^* \to A^*$ are bijectives(by definition 2.3.9)

Now since $f^{-1}: B^* \to A^*$ is bijective, we have

$(f^{-1})^{-1}: A^* \to B^*$ also bijective (proposition 3.1)

Therefore $(f^{-1})^{-1} = f$.

Hence the result.

Proposition 3.10: Let $A, B, C \in M(X)$. If $f: A \to B$ and $g: B \to C$ are constant mset functions, then gof is a constant mset function.

Proof: We need to show that $gof: A \to C$ is a constant mset function.

Now $f: A^* \to B^*$ and $g: B^* \to C^*$ are constant functions (by hypothesis)

In particular $gof: A^* \to C^*$ is constant.

Thus gof is a constant mset function (by definition 2.3.3).

Proposition 3.11: Let $A, B \in M(X)$, such that the mset function $f: A \to B$ is bijective, then $f: A^n \to B^n$ is bijective for $n \in \{0,1,2, \dots\}$

Proof: Let $f: A \to B$ be a bijective mset function, then $f: A^* \to B^*$ is bijective (i) and

$\forall x (x \in A^* \Rightarrow C_A(x) = C_B(f(x)))$ (ii)

We are to show that $f: A^n \to B^n$ is bijective for $n \in \{0,1,2, \dots\}$

Now from proposition 2.2.10, $A^* = (A^n)^*$. Similarly $B^* = (B^n)^*$ and hence

$f: (A^n)^* \to (B^n)^*$ is just the function $f: A^* \to B^*$. (iii)

Thus $f: (A^n)^* \to (B^n)^*$ is bijective since $f: A^* \to B^*$ is bijective. (iv)

Now we show that $\forall x (x \in (A^n)^* = A^* \Rightarrow C_{A^n}(x) = C_{B^n}(f(x)))$

Also from (ii) and definition 2.2.9 $C_{A^n}(x) = (C_A(x))^n = (C_B(f(x)))^n = C_{B^n}(f(x))$.

In particular $C_{A^n}(x) = C_{B^n}(f(x))$ (v)

It is clear from (iv) and (v) above that

$f: A^n \to B^n$ is bijective for $n \in \{0,1,2, \dots\}$

Proposition 3.12: Let $A, B \in M(X)$, then the similarity relation $\sim$ is an equivalence relation.

Proof: Supposing $A \in M(X)$, then $A \sim A$ via the identity mset function (proposition 3.6). Thus $\sim$ is reflexive on $M(X)$.

Also, given that $A, B \in M(X)$, then the similarity relation $A \sim B \Rightarrow B \sim A$ via the inverse of a bijective mset function (proposition 3.1). Thus $\sim$ is symmetric on $M(X)$.

Again, suppose that $A, B, C \in M(X)$ with $A \sim B$ and $B \sim C$, then $A \sim C$ via the composition of bijective mset functions (proposition 3.5). Thus $\sim$ is transitive on $M(X)$.

In particular $\sim$ is an equivalence relation on $M(X)$.

Proposition 3.13: Let $A, B \in M(X)$. If $A \preccurlyeq B$ and that $B \preccurlyeq C$ .Then $A \preccurlyeq C$.

Proof: Clearly $A \preccurlyeq C$. (From Definition 2.3.14 and proposition 3.3).

Proposition 3.14: Let $A, B \in M(X)$. Then $A \sim B \Rightarrow A \preccurlyeq B$.

Proof: Supposing $A \sim B$.

Let $f: A \to B$ be a bijection (by definition 2.3.9), then $f: A \to B$ is both an injection and surjection(by definition 2.3.9).

Thus $A \preccurlyeq B$. (by definition 2.3.13).

In particular, $A \sim B \Rightarrow A \preccurlyeq B$.

Proposition 3.15: Let , $B \in M(X)$. If $A \sim B$, then $A \preccurlyeq B$ and $B \preccurlyeq A$.

Proof: Supposing $A \sim B$. Then $B \sim A$.(from proposition 3.12)

$A \sim B \Rightarrow A \preccurlyeq B$ and $B \sim A \Rightarrow B \preccurlyeq A$ (from proposition 3.13).

However, the converse need not hold For example, let $A = \{x_1, x_2, x_3 \ldots\}_{2,4,6,\ldots}$, and $B = \{y_0, y_1, x_2 \ldots\}_{1,3,5,\ldots}$ be msets.

The function $f: A^* \to B^*$ defined by $f(x_n) = y_n$ makes $f: A \to B$ an injection so that $A \preccurlyeq B$.

Also the function $g: B^* \to A^*$ defined by $g(y_n) = x_{n+1}$ makes $g: B \to A$, the inverse mset function, an injection so that $B \preccurlyeq A$.

However, there cannot be a bijection $h: A \to B$ since all multiplicities in A are even and those in B are odd.

4. Conclusions.

We have introduced mset functions in a unique way. We have shown that an mset function is bijective if and only if its inverse function is bijective. It has also been established that the composition of injective, surjective and bijective mst functions is injective, surjective and bijective respectively. We have also shown that the identity mset function is bijective. so also the composition of constant msets function is also constant. We have shown that the bijectiveness of mset function is preserved on raising to arithmetic power of msets. similarity relations was introduced and shown that it is an equivalence relation and that similarity relation implies dominance relation. However, Schroider Bernstein theorem for mset functions fails and consequently the dominance relation cannot be a partial order relative to similarity relation on the class $M(X)$.

Further research direction

The definition of countability of an mset as well as the study of equivalent classes of similarity relation in $M(X)$ looks promising and challenging.

References.

[1] Blizard, W..(1989); Multiset Theory, Notre Dame, J. Formal Logic 30, 36-66.

[2] Cerf V., Fenandez E., Gostelow K. and Volausky S., Formal control and low properties of a model of computation, Report ENG 7178, Computer Science Department, University of California, Los Angeles, CA, December, pp. 81, (1971).

[3] Lake, J. (1976), Sets, Fuzzy Sets, Multisets and Functions. Journal of London Mathematical Society (2), 12: 323-326.

[4] Knuth, D.E. (1981); The art of computer programming, Vol. 2 (semi numerical Algorithms), Second Edition, Addison-Wesley, Reading, Massachusetts.

[5] Silvia, G., Pantovic, J and Gradimir, V. (2014). Binary Relations and Algebras on Multisets. Publications de L'Institute Mathematique, Nouvelle serie, tome 95 (109), 111-117

[6] Levy, A.,(1997); Basic set theory, Springer-Verlag, Berlin.

[7] Parker – Rhodes, A.F. (1981); The Theory of Indistinguishables, D. Reidel, Dordrecht.

[8] Petrovsky, A. B, (1997); Structuring Techniques in Multiset Spaces. Springer. 174-184

[9] Wildberger, N.J. (2003); A new look at multiset, School of Mathematics, UNSW, Sydney.

[10] Shravan K. and Tripathy B.C., (2018); Generalised closed Sets in multiset topological spaces, Proyecciones Jour. Math. Vol. 37, 2.pp. 223-237.

[11] Singh, D., Ibrahim, A,M., Yohanna, T. and Singh, J.N. (2011) – Complemention in multiset Theory, International Mathematica Forum, 6(38), 1877.

[12] K.P. Girish and Sunil Jacob John, (2009); Relatioins and Functions in Multiset Context; Information Sciences 179(2009),756-768.

[13] Hallet, M (1984). Cantorian Set Theory and Limitation of Size, Clarendon Press, Oxford.

[14] Hickman, J. L. (1980)@ A Note on the Concept of Multiset. A bulletin of the Austrailian Mathematical Society, 22: 211-217.

[15] Yager, R. R, (1986); On the theory of bags. International Journal of Genaral System, 13; 23-37

[16] Tero Harju, (1996); Lecture notes on Semigroups, Department of Mathematics, University of Turku, FIN-20014 Turku, Finland.

[17] K.P. Girish and Sunil Jacob John, (2012); On Multiset Topologies; Theory and Application of Mathematical and Computer Sciences (2)(2012),37-52